

# MULTIDIMENSIONAL BORG–LEVINSON THEOREMS FOR UNBOUNDED POTENTIALS

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ABSTRACT. We prove that the Dirichlet eigenvalues and Neumann boundary data of the corresponding eigenfunctions of the operator  $-\Delta + q$ , determine the potential  $q$ , when  $q \in L^{n/2}(\Omega, \mathbb{R})$  and  $n \geq 3$ . We also consider the case of incomplete spectral data, in the sense that the above spectral data is unknown for some finite number of eigenvalues. In this case we prove that the potential  $q$  is uniquely determined for  $q \in L^p(\Omega, \mathbb{R})$  with  $p = n/2$ , for  $n \geq 4$  and  $p > n/2$ , for  $n = 3$ .

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$  be a bounded domain, with a smooth boundary. The operator  $-\Delta + q$ , with  $q \in L^{n/2}(\Omega, \mathbb{R})$  and form domain  $H_0^1(\Omega)$ , has a spectrum consisting of a discrete set of real eigenvalues,  $\lambda_k$  of finite multiplicity, such that

$$-\infty < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow \infty,$$

as  $k \rightarrow \infty$ . The eigenvalues correspond to eigenfunctions  $\varphi_k$ , which form an orthonormal basis in  $L^2(\Omega)$  (see the appendix in section 5 for some further discussion).

The multi-dimensional Borg-Levinson problem first considered in [17], by Nachman, Sylvester and Uhlmann and independently by Novikov in [18], is an inverse spectral problem that asks if a potential  $q$  is uniquely determined if one knows

$$(1.1) \quad \lambda_k \text{ and } \nu \cdot \nabla \varphi_k|_{\partial\Omega}, \text{ for } k \in \mathbb{N},$$

where  $\nu$  is the outward pointing unit normal vector to the boundary  $\partial\Omega$ . Nachman, Sylvester and Uhlmann showed that this is indeed possible for  $q \in L^\infty(\Omega, \mathbb{R})$ . This result is a higher dimensional variant of a question studied originally by Borg in [3] and Levinson in [15], in the case of the 1-dimensional Schrödinger equation.

The multi-dimensional Borg-Levinson problem has been studied in a number of settings. It is not possible to give an extensive survey of this here and we will only mention a few results which are of relevance here.

The case unbounded or singular potentials  $q \in L^p(\Omega, \mathbb{R})$   $p > n/2$ , has been studied by Päivärinta and Serov in [19]. Krupchyk and Päivärinta

studied the problem for higher order elliptic operators in [12], in the case that  $q \in L^\infty(\Omega, \mathbb{R})$ .

The problem has also been considered in the case when the spectral data for a finite number of the eigenvalues is unknown. One of the first to study the case of incomplete spectral data was Isozaki in [10]. A further interesting development in this direction is the work by Choulli and Stefanov in [6] where they show that one only needs assume that the spectral data is asymptotically near to each other, to obtain uniqueness.

The main results here are the following. We use the notation  $\lambda_{q_j,k}$  and  $\varphi_{q_j,k}$  for the  $k$ th eigenvalue and eigenvector corresponding to the operator  $-\Delta + q_j$ . And the notation  $\tilde{\gamma}u$  for the trace of the normal derivative of  $u$  to  $\partial\Omega$ , which corresponds to  $\nu \cdot \nabla u|_{\partial\Omega}$  when  $u$  is smooth (see section 6 for further details).

**Theorem 1.1.** *Suppose that  $q_1, q_2 \in L^{n/2}(\Omega, \mathbb{R})$ , with  $n \geq 3$  and that*

$$\lambda_{q_1,k} = \lambda_{q_2,k} \quad \text{and} \quad \tilde{\gamma}\varphi_{q_1,k} = \tilde{\gamma}\varphi_{q_2,k},$$

*for  $k \in \mathbb{N}$ , then  $q_1 = q_2$ .*

In the case of incomplete spectral data we have.

**Theorem 1.2.** *Let  $q_1, q_2 \in L^p(\Omega, \mathbb{R})$ , where  $p = n/2$ , when  $n \geq 4$  and  $p > n/2$ , when  $n = 3$ , and suppose that there is a  $k_0 \in \mathbb{N}$  such that*

$$\lambda_{q_1,k} = \lambda_{q_2,k} \quad \text{and} \quad \tilde{\gamma}\varphi_{q_1,k} = \tilde{\gamma}\varphi_{q_2,k},$$

*for  $k \geq k_0$ , then  $q_1 = q_2$ .*

The above Theorems improve the results in [19] in two ways when  $n \geq 3$ . We firstly prove the Borg-Levinson Theorem for singular or unbounded potentials in the limiting case of  $q \in L^{n/2}(\Omega, \mathbb{R})$ . The second Theorem extends the result in [19] to the case of incomplete spectral data, when  $q \in L^p(\Omega, \mathbb{R})$  and  $p > n/2$ , if  $n = 3$  and  $p = n/2$ , if  $n \geq 4$ . We do not consider the two dimensional case here. The character of the two dimensional problem is somewhat different, since in this case  $q \in L^1(\Omega, \mathbb{R})$ .

Theorem 1.1 is proved by reducing it to the corresponding inverse boundary value problem, which has been solved for  $q \in L^{n/2}(\Omega)$  (see [14], [4] and [8]). The proof here is roughly of the same form as the argument in [5]. Here we however use the  $L^p$ -theory of elliptic equations in a fairly systematic way, which enables us to handle unbounded potentials.

The proof of Theorem 1.2 is based on the argument used in [10]. Here one needs to consider a spectral parameter that goes to infinity in a specific way. One needs moreover some form of  $L^p$  resolvent estimates with an explicit dependence on the spectral parameter. The most interesting case here is  $n = 3$ , since  $n/2 = 3/2 < 2$ . It turns

out that for  $q \in L^p(\Omega)$ , with  $p > n/2$ , one can still use  $L^2$ -theory and interpolation, to prove Theorem 1.2. The case  $p = 3/2$  seems however to require better estimates, where the spectral parameter  $\lambda \in \mathbb{C}$  is allowed, to grow more freely (than in e.g. Proposition 2.5), similar to the so called uniform  $L^p$ -estimates found in [11] and [8],

The paper is structured as follows. In section 2 we prove that the Dirichlet problem for  $-\Delta + q$  admits strong solutions, when considering appropriate boundary data. We also derive some a-priori  $L^p$ -estimates. In section 3 we prove Theorem 1.1. In section 4 we consider the case of incomplete spectral data and prove Theorem 1.2. At the end of the paper we have included two appendices. The first one in section 5 reviews some facts from spectral theory that we use. The second appendix in section 6 presents some basic facts concerning Besov spaces.

## 2. A PRIORI ESTIMATES AND STRONG SOLUTIONS

The aim of this section is to show in some detail that the Dirichlet problem for  $-\Delta + q - \lambda$ ,  $q \in L^{n/2}(\Omega, \mathbb{R})$  admits strong solutions when considering boundary conditions from appropriate spaces. Recall that a strong solution is a solution in  $W^{2,p}(\Omega)$ , which satisfies the equation almost everywhere. Having strong solutions will guarantee that we can later define the so called Dirichlet-to-Neumann map in a suitable way. We end the section by deriving an a priori estimate that has an explicit dependence on the spectral parameter  $\lambda$ .

Our main aim will be to prove existence and uniqueness of solutions to the Dirichlet problem

$$(2.1) \quad \begin{aligned} (-\Delta + q - \lambda)u &= 0 & \text{in } \Omega, \\ \gamma u &= f & \text{on } \partial\Omega, \end{aligned}$$

when  $q \in L^{n/2}(\Omega, \mathbb{R})$  and  $f \in B_{pp}^{2-1/p}(\partial\Omega)$ .

We begin by deriving solutions to the corresponding inhomogeneous problem with zero Dirichlet condition. To this end we will need a priori estimates in  $L^p$ -norms, which we now state.

**Proposition 2.1.** *Assume  $q \in L^{n/2}(\Omega)$ . Suppose  $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ ,  $p = \frac{2n}{n+2}$ , then*

$$(2.2) \quad \|u\|_{W^{2,p}(\Omega)} \leq C(\|(-\Delta + q)u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}),$$

*Proof.* By the estimate (2.12) of Lemma 2.5 at the end of this section, we have that

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega)} &\leq C_0 \|(-\Delta + q - \lambda_0)u\|_{L^p(\Omega)} \\ &\leq C(\|(-\Delta + q)u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}) \end{aligned}$$

for some  $\lambda_0 \in \mathbb{R}_-$ .

□

The previous a priori estimate and the resolvent estimate (5.3) for Sobolev spaces can be used to prove the following existence and uniqueness result or the inhomogeneous problem, when  $q \in L^{n/2}(\Omega)$  (see also Lemma 9.17 and Theorem 9.15 in [9]). Notice also that the Proposition applies to complex  $\lambda$ .

**Proposition 2.2.** *Let  $q \in L^{n/2}(\Omega)$  and  $p = 2n/(n+2)$ . There exists a  $\lambda_0 \in \mathbb{R}$  such that for  $\lambda \in \mathbb{C} \setminus (\lambda_0, \infty)$ , there is a unique strong solution  $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  to*

$$(2.3) \quad \begin{aligned} (-\Delta + q - \lambda)w &= F & \text{in } \Omega, \\ \gamma w &= 0 & \text{on } \partial\Omega, \end{aligned}$$

for all  $F \in L^p(\Omega)$ . Moreover we have that

$$(2.4) \quad \|w\|_{W^{2,p}(\Omega)} \leq C_\lambda \|F\|_{L^p(\Omega)}.$$

*Proof.* Pick  $F_k \in L^2(\Omega)$ , s.t.  $F_k \rightarrow F$ , in the  $L^p(\Omega)$ -norm. Denote by  $u_k$  the corresponding solutions to the problem

$$\begin{aligned} (-\Delta + q - \lambda)u_k &= F_k & \text{in } \Omega, \\ \gamma u_k &= 0 & \text{on } \partial\Omega, \end{aligned}$$

From the  $L^2$ -theory of elliptic partial differential operators, we know that there is an  $\lambda_0$  such that the solution  $u_k$  exists and is unique, when  $\lambda \in \mathbb{C} \setminus (\lambda_0, \infty)$ . By (2.2) we have the estimate

$$\|u_k\|_{W^{2,p}(\Omega)} \leq C_\lambda (\|(-\Delta + q - \lambda)u_k\|_{L^p(\Omega)} + \|u_k\|_{L^p(\Omega)}).$$

The solution is given by the resolvent, i.e.  $u_k = R_q(\lambda)F_k \in H_0^1(\Omega)$ . For the resolvent we have estimate (5.3) and by Sobolev embedding we have that

$$\|R_q(\lambda)F_k\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C_\lambda \|F_k\|_{L^{\frac{2n}{n+2}}(\Omega)}.$$

Combining the two previous estimates we get that

$$(2.5) \quad \begin{aligned} \|u_k\|_{W^{2,p}(\Omega)} &\leq C_\lambda (\|F_k\|_{L^p(\Omega)} + \|R_q(\lambda)F_k\|_{L^p(\Omega)}) \\ &\leq C_\lambda \|F_k\|_{L^p(\Omega)} < C_\lambda < \infty \end{aligned}$$

Now  $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset W^{2,p}(\Omega)$  is a complete and reflexive subspace, and  $\{u_k\}$  is a bounded set in this subspace because of (2.5). As a consequence we obtain a subsequence  $w_k$  that converges weakly, i.e.  $w_k \rightharpoonup w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ . The weak convergence implies that

$$\int_{\Omega} \varphi D^\alpha w_k \rightarrow \int_{\Omega} \varphi D^\alpha w,$$

for multi-indices  $\alpha$ ,  $|\alpha| \leq 2$  and  $\varphi \in L^{2n/(n-2)}(\Omega)$ . This implies that

$$(2.6) \quad \int_{\Omega} \nabla \varphi \cdot \nabla (w_k - w) \varphi \rightarrow 0,$$

when  $\varphi \in C_0^\infty(\Omega)$ . The Rellich-Kondrachov Theorem implies on the other hand that the embedding  $id: W^{2,p}(\Omega) \rightarrow L^q(\Omega)$  is compact when  $q < 2n/(n-2)$ , so that  $w_k \rightarrow w$  in the  $L^q(\Omega)$ -norm, for  $q < 2n/(n-2)$ . It follows in particular that  $w_k \rightarrow w$  in the  $L^{n/(n-2)}(\Omega)$ -norm. This and the Hölder inequality gives that

$$(2.7) \quad \int_{\Omega} (q - \lambda)(w_k - w)\varphi \leq \|(q - \lambda)\varphi\|_{L^{\frac{n}{2}}(\Omega)} \|w_k - w\|_{L^{\frac{n}{n-2}}(\Omega)} \rightarrow 0,$$

for  $\varphi \in C_0^\infty(\Omega)$ . It is straight forward to see using (2.6), (2.7) and that  $F_k \rightarrow F$ , in the  $L^p(\Omega)$ -norm, that  $w$  is a weak solution to (2.3). And thus  $w$  a strong solution, since  $w \in W^{2,p}(\Omega)$ .

The weak solution  $w$  is in  $H_0^1(\Omega)$  and is a weak solution to (2.3), when  $F \in L^p(\Omega)$  is taken as an element in  $H^{-1}(\Omega)$ . The  $L^2$ -theory of elliptic operators implies that  $w$  is the unique solution in  $H_0^1(\Omega)$ . It follows that  $w$  is the unique solution in  $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ .

It remains to prove the norm estimate. Assume that estimate (2.4) is false, then there exists a sequences of functions  $F_k \in L^p(\Omega)$  and corresponding solutions  $u_k \in W^{2,p}(\Omega)$  s.t.

$$\|u_k\|_{L^p(\Omega)} \geq k \|F_k\|_{L^p(\Omega)}.$$

We may assume that  $\|u_k\|_{L^p(\Omega)} = 1$ , so that  $F_k \rightarrow 0$ , in the  $L^p(\Omega)$ -norm. Using (2.2) again, we have the estimate

$$\|u_k\|_{W^{2,p}(\Omega)} \leq C_\lambda (\|F_k\|_{L^p(\Omega)} + \|u_k\|_{L^p(\Omega)}) < M < \infty.$$

The set  $\{u_k\}$  is hence bounded in the subspace  $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ . As earlier in the proof there is a subsequence  $w_k$  that converges weakly, i.e.  $w_k \rightharpoonup w_0 \in W^{2,p}(\Omega)$  in the  $W^{2,p}(\Omega)$ -norm. The Rellich-Kondrachov Theorem implies again that  $w_k \rightarrow w_0$  in the  $L^q(\Omega)$ -norm, for  $q < 2n/(n-2)$ . Which implies that  $\|w_0\|_{L^p(\Omega)} = 1$ , since we picked  $\|u_k\|_{L^p(\Omega)} = 1$ . Arguing as in the first part of the proof, we see that  $w_0$  solves

$$\begin{aligned} (-\Delta + q - \lambda)w_0 &= 0 \quad \text{in } \Omega, \\ \gamma w_0 &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The solution to (2.3) is however unique and thus  $w_0 \equiv 0$ , which is a contradiction, since  $\|w_0\|_{L^p(\Omega)} = 1$ . □

Recall that we can reduce the problem in (2.1) to the inhomogeneous problem in (2.3). The proof of the following Lemma is standard. We give it here as a convenience and since the argument has a dependence on  $\lambda$ .

**Lemma 2.3.** *The boundary value problem (2.1) has a unique solution  $u \in W^{2,p}(\Omega)$ ,  $p = 2n/(n+2)$  if the inhomogeneous problem*

$$(2.8) \quad \begin{aligned} (-\Delta + q - \lambda)w &= F \quad \text{in } \Omega, \\ \gamma w &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

admits a unique solution  $w \in W^{2,p}(\Omega)$ , for all  $F \in L^p(\Omega)$ . If moreover the estimate

$$(2.9) \quad \|w\|_{W^{2,p}(\Omega)} \leq C\|F\|_{L^p(\Omega)},$$

holds, then we have the estimate

$$(2.10) \quad \|u\|_{W^{2,p}(\Omega)} \leq C_\lambda \|f\|_{B_{pp}^{2-1/p}(\partial\Omega)}.$$

*Proof.* Choose  $F$  so that  $F = (-\Delta + q - \lambda)Ef$ , where  $E$  is the extension operator defined in section 6. Notice that  $F \in L^p(\Omega)$ , because

$$\|\Delta Ef\|_{L^p(\Omega)} \leq C\|Ef\|_{W^{2,p}(\Omega)},$$

and since by Sobolev embedding  $W^{2,p}(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$  and the Hölder inequality we have that

$$\begin{aligned} \|qEf\|_{L^{\frac{2n}{n+2}}(\Omega)} &\leq \|q\|_{L^{\frac{n}{2}}(\Omega)} \|Ef\|_{L^{\frac{2n}{n-2}}(\Omega)} \\ &\leq \|q\|_{L^{\frac{n}{2}}(\Omega)} \|Ef\|_{W^{2,p}(\Omega)}. \end{aligned}$$

Let  $w \in W^{2,p}(\Omega)$  be the unique solution of (2.8), corresponding to  $-F$ . The function  $u := w + Ef \in W^{2,p}(\Omega)$  solves then (2.1). This proves existence.

Suppose on the other hand that  $u_1$  and  $u_2$  solve (2.1). Then  $u_1 - u_2$  will be a solution to (2.8) with a zero source term. Uniqueness for (2.8) now implies that  $u_1 = u_2$ .

To obtain the norm estimate (2.10) we use (2.9) and argue as follows.

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega)} &\leq \|w\|_{W^{2,p}(\Omega)} + \|Ef\|_{W^{2,p}(\Omega)} \\ &\leq C\|(-\Delta + q - \lambda)Ef\|_{L^p(\Omega)} + \|Ef\|_{W^{2,p}(\Omega)} \\ &\leq C_\lambda \|Ef\|_{W^{2,p}(\Omega)} \\ &\leq C_\lambda \|f\|_{B_{pp}^{2-1/p}(\partial\Omega)} \end{aligned}$$

□

As a Corollary to Lemma 2.3 and Proposition 2.2 we have the following existence result.

**Corollary 2.4.** *Let  $q \in L^{n/2}(\Omega)$  and  $p = 2n/(n+2)$ . There exists a  $\lambda_0 \in \mathbb{R}$ , such that the Dirichlet problem (2.1) has a unique strong solution  $u \in W^{2,p}(\Omega)$ , when  $\lambda \in \mathbb{C} \setminus (\lambda_0, \infty)$ . Moreover*

$$(2.11) \quad \|u\|_{W^{2,p}(\Omega)} \leq C_\lambda \|f\|_{B_{pp}^{2-1/p}(\partial\Omega)}.$$

We end this section by formulating an  $L^p$ -apriori estimate which has an explicit dependence on  $\lambda$ , when  $\lambda \in \mathbb{R}$ . This type of estimate was derived by Agmon in [1] (see Theorem 2.1.). We need to modify it by adding a  $q \in L^{n/2}(\Omega)$  to the operator. This estimate is one of our main tools.

**Proposition 2.5.** Assume  $\frac{2n}{n+2} \leq p < \infty$  and that  $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ . Then we have that

$$(2.12) \quad \sum_{j=0}^2 |\lambda|^{\frac{2-j}{2}} \|u\|_{W^{j,p}(\Omega)} \leq C \|(-\Delta + q - \lambda)u\|_{L^p(\Omega)},$$

when  $-\lambda \in \mathbb{R}$  is large. The constant  $C$  does not depend on  $\lambda$ .

*Proof.* We begin by choosing a  $\mu \in \mathbb{R}_+$  s.t.  $\mu^2 = -\lambda$ . Let  $D = \Omega \times (-1, 1)$  and let  $\zeta \in C_0^\infty(-1, 1)$  be s.t.  $\zeta(t) = 1$  when  $|t| \leq 1/2$ . Define  $v(x, t) := \zeta(t)e^{i\mu t}u(x)$  and  $\tilde{q}(x, t) := q(x)$ .

By Theorem 2.1 and its proof in [1] (see also [20]) there is a  $\lambda_0$  s.t.

$$\|(-\Delta_x - \partial_t^2 - \lambda_0)v\|_{L^p(D)} \geq C_0 \|v\|_{W^{2,p}(D)}.$$

It follows that

$$\begin{aligned} \|(-\Delta_x - \partial_t^2 + \tilde{q})v\|_{L^p(D)} &\geq \|(-\Delta_x - \partial_t^2 - \lambda_0)v\|_{L^p(D)} - \|\tilde{q}v\|_{L^p(D)} \\ &\quad - |\lambda_0| \|v\|_{L^p(D)} \\ &\geq C_0 \|v\|_{W^{2,p}(D)} - \|\tilde{q}v\|_{L^p(D)} - |\lambda_0| \|v\|_{L^p(D)}. \end{aligned}$$

Let  $\tilde{q}_k \in C_0^\infty(D)$  be s.t.  $\tilde{q}_k \rightarrow \tilde{q}$  in the  $L^{n/2}(D)$ -norm. Then by the Hölder inequality and Sobolev embedding we have that

$$\begin{aligned} \|\tilde{q}v\|_{L^p(D)} &\leq \|\tilde{q} - \tilde{q}_k\|_{L^{n/2}(D)} \|v\|_{L^{\frac{2n}{n-2}}(D)} + \|\tilde{q}_k\|_{L^\infty(D)} \|v\|_{L^p(D)} \\ &\leq \frac{C_0}{2} \|v\|_{W^{2,p}(D)} + C_k \|v\|_{L^p(D)}, \end{aligned}$$

when  $k$  is chosen to be large enough. Combining the two previous estimates gives that

$$\|(-\Delta_x - \partial_t^2 + \tilde{q})v\|_{L^p(D)} \geq \frac{C_0}{2} \|v\|_{W^{2,p}(D)} - (C_k + |\lambda_0|) \|v\|_{L^p(D)}.$$

Or in other words that

$$(2.13) \quad \|v\|_{W^{2,p}(D)} \leq C (\|(-\Delta_x - \partial_t^2 + \tilde{q})v\|_{L^p(D)} + \|v\|_{L^p(D)}),$$

for  $p \geq 2n/(n+2)$ . Using the definition of  $v$  we have that

$$(-\Delta_x - \partial_t^2)v = (-\zeta\Delta_x u - (\zeta'' - \zeta\mu^2 + 2i\mu\zeta')u)e^{i\mu t},$$

We can thus estimate the first term on the right hand side of (2.13) as

$$\|(-\Delta_x - \partial_t^2 + \tilde{q})v\|_{L^p(D)} \leq C (\|(-\Delta_x + q + \mu^2)u\|_{L^p(\Omega)} + (1 + |\mu|) \|u\|_{L^p(\Omega)}).$$

Next consider the region  $D' = \Omega \times (-\frac{1}{2}, \frac{1}{2})$ . Since  $\zeta = 1$  in  $D'$ , we can write the Sobolev norm of  $v$  as

$$\begin{aligned} \|v\|_{W^{2,p}(D')} &= (1 + |\mu| + |\mu|^2) \|u\|_{L^p(\Omega)} \\ &\quad + \sum_{j=1}^n (1 + |\mu|) \|\partial_j u\|_{L^p(\Omega)} + \sum_{j \leq i}^n \|\partial_i \partial_j u\|_{L^p(\Omega)}. \end{aligned}$$

Estimating the left hand side of (2.13) from below, by taking the  $L^p(D')$ -norm and temporarily dropping the  $L^p$ -norms of the derivatives in the above expression, gives

$$|\mu|^2 \|u\|_{L^p(\Omega)} \leq C \left( \|(-\Delta_x + q + \mu^2)u\|_{L^p(\Omega)} + (2 + |\mu|) \|u\|_{L^p(\Omega)} \right).$$

We can absorb the second term on the right hand side by the first term by picking a large  $|\mu|$ . We thus get that

$$|\mu|^2 \|u\|_{L^p(\Omega)} \leq C \|(-\Delta_x + q - \lambda)u\|_{L^p(\Omega)}.$$

By adding back the  $\|\partial_j u\|_{L^p(\Omega)}$  and the  $\|\partial_k \partial_j u\|_{L^p(\Omega)}$ , that we dropped from the left hand side, yields

$$|\mu|^2 \|u\|_{L^p(\Omega)} + |\mu| \|\partial_j u\|_{L^p(\Omega)} + \|\partial_i \partial_j u\|_{L^p(\Omega)} \leq C \|(-\Delta_x + q - \lambda)u\|_{L^p(\Omega)},$$

which implies the estimate of the claim.  $\square$

Note that the above argument can be extended so that it applies to  $\lambda$  that lie on certain rays emanating from the origin in  $\mathbb{C}$  (see [1]).

### 3. FROM SPECTRAL DATA TO THE DIRICHLET-TO-NEUMANN MAPS

In this section we show that the spectral data determines the Dirichlet-to-Neumann map  $\Lambda_q(\lambda)$ , when  $-\lambda \in \mathbb{R}$  is large. This in conjunction with known results on the inverse boundary problem, will provide a proof for Theorem 1.1, which we spell out at the end of the section.

We begin by defining the Dirichlet-to-Neumann map. Let  $q \in L^{\frac{n}{2}}(\Omega, \mathbb{R})$ ,  $n \geq 3$ . Consider the Dirichlet problem

$$(3.1) \quad \begin{aligned} (-\Delta + q - \lambda)u &= 0, & \text{in } \Omega, \\ u|_{\partial\Omega} &= f, & \text{on } \partial\Omega, \end{aligned}$$

where

$$f \in B_{pp}^{2-1/p}(\partial\Omega),$$

with  $p = 2n/(n+2)$ . The boundary value problem (3.1) has a unique solution  $u \in W^{2,p}(\Omega)$  due to Lemma 2.4, when  $-\lambda$  is large. The solution  $u$  is furthermore bounded by the Dirichlet data, i.e.

$$(3.2) \quad \|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{B_{pp}^{2-1/p}(\partial\Omega)}.$$

Recall that  $\tilde{\gamma}$  is the Neumann trace operator, which gives meaning to the restriction  $\nu \cdot \nabla u|_{\partial\Omega}$  when  $u$  is non-smooth. We define the map  $\Lambda_q$  as

$$\Lambda_q(\lambda)f := \tilde{\gamma}u,$$

where  $u$  is the unique solution to (3.1) with boundary data  $f$ . The map  $\Lambda_q$  is called the Dirichlet-to-Neumann map or the DN-map for short. Estimate (3.2) and the continuity of the Neumann trace operator, shows that  $\Lambda_q(\lambda) : B_{pp}^{2-1/p}(\partial\Omega) \rightarrow B_{pp}^{1-1/p}(\partial\Omega)$  is continuous.



One of the main concerns of this section is to investigate the difference  $[\Lambda_{q_1}(\lambda) - \Lambda_{q_2}(\lambda)]f$ , as  $\lambda \rightarrow -\infty$ . In considering the DN-maps we will need an estimate with an explicit dependence on  $\lambda$  for the homogeneous problem (3.1). To this end it will be convenient to state the following Lemma, which is a direct consequence of Proposition 2.5.

**Lemma 3.1.** *Let  $w \in W^{2,p}(\Omega)$ ,  $p = 2n/(n+2)$  be a solution to (2.8). Then there is a constant  $C$  independent of  $\lambda$ , for large  $-\lambda \in \mathbb{R}$ , such that*

$$(3.3) \quad \|w\|_{L^p(\Omega)} \leq C|\lambda|^{-1}\|F\|_{L^p(\Omega)},$$

$$(3.4) \quad \|w\|_{W^{2,p}(\Omega)} \leq C\|F\|_{L^p(\Omega)}.$$

The estimate of the next Lemma is similar to the estimate of Lemma 2.4. Here we need the constant in the estimate to be independent of  $\lambda$ , which requires some additional effort.

**Lemma 3.2.** *Suppose  $u \in W^{2,p}(\Omega)$ ,  $p = 2n/(n+2)$  solves (3.1). Then there exists a constant  $C$  independent of  $\lambda$ , such that*

$$(3.5) \quad \|u\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C\|f\|_{B_{pp}^{2-1/p}(\partial\Omega)}.$$

*Proof.* Decompose  $u$  as  $u = v_0 + v_1$  where

$$(3.6) \quad \begin{aligned} (-\Delta - \lambda)v_0 &= 0 & \text{in } \Omega, \\ \gamma v_0 &= f & \text{on } \partial\Omega. \end{aligned}$$

And

$$(3.7) \quad \begin{aligned} (-\Delta + q - \lambda)v_1 &= -qv_0 & \text{in } \Omega, \\ \gamma v_1 &= 0 & \text{on } \partial\Omega. \end{aligned}$$

By estimate (3.4) in Lemma 3.1 and Sobolev embedding we have that

$$\begin{aligned} \|v_1\|_{L^{\frac{2n}{n-2}}(\Omega)} &\leq C\|v_1\|_{W^{2,p}(\Omega)} \leq C\|qv_0\|_{L^{\frac{2n}{n+2}}(\Omega)} \\ &\leq C\|q\|_{\frac{n}{2}(\Omega)}\|v_0\|_{L^{\frac{2n}{n-2}}(\Omega)}, \end{aligned}$$

where  $C$ , independent of  $\lambda$ , for large  $-\lambda$ . It is therefore enough to obtain a bound on the  $L^p$ -norm for  $v_0$ , with  $p = 2n/(n-2)$  with a constant that is independent of  $\lambda$ . We do this by making a second splitting. We set  $v_0 = w_0 + w_1$ , where

$$(3.8) \quad \begin{aligned} (-\Delta - \tilde{\lambda})w_0 &= 0 & \text{in } \Omega, \\ \gamma w_0 &= f & \text{on } \partial\Omega. \end{aligned}$$

where  $\tilde{\lambda}$  is chosen with  $-\tilde{\lambda}$  so large that the problem has a unique solution. The function  $w_1$  solves

$$(3.9) \quad \begin{aligned} (-\Delta - \lambda)w_1 &= (\lambda - \tilde{\lambda})w_0 & \text{in } \Omega, \\ \gamma w_1 &= 0 & \text{on } \partial\Omega. \end{aligned}$$

We begin by estimating  $w_0$ , with writing it by means of an inhomogeneous problem, i.e. we set  $w_0 = \tilde{w} + Ef$ , where  $\tilde{w}$  is the unique solution to

$$(3.10) \quad \begin{aligned} (-\Delta - \tilde{\lambda})\tilde{w} &= (\Delta + \tilde{\lambda})Ef \quad \text{in } \Omega, \\ \gamma\tilde{w} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

For  $\tilde{w}$  we have by Proposition 2.2 and the continuity of right inverse of the trace operator, that

$$\begin{aligned} \|\tilde{w}\|_{W^{2,p}(\Omega)} &\leq C\|(\Delta + \tilde{\lambda})Ef\|_{L^p(\Omega)} \\ &\leq C\|(\Delta + \tilde{\lambda})Ef\|_{L^p(\Omega)} \\ &\leq C\|Ef\|_{W^{2,p}(\Omega)} + |\tilde{\lambda}|\|Ef\|_{L^p(\Omega)} \\ &\leq C\|f\|_{B_{pp}^{2-1/p}(\partial\Omega)}. \end{aligned}$$

Notice that the constant  $C$  is independent of  $\lambda$ . The above gives with Sobolev embedding the estimate

$$\begin{aligned} \|w_0\|_{L^{\frac{2n}{n-2}}(\Omega)} &\leq C\|w_0\|_{W^{2,p}(\Omega)} \leq C\|\tilde{w}\|_{W^{2,p}(\Omega)} + \|Ef\|_{L^p(\Omega)} \\ &\leq C\|f\|_{B_{pp}^{2-1/p}(\partial\Omega)}. \end{aligned}$$

Now we can apply (2.12) with (3.9) in the case  $p = 2n/(n-2)$ . This gives

$$\begin{aligned} \|w_1\|_{L^{\frac{2n}{n-2}}(\Omega)} &\leq \frac{C}{|\lambda|}\|(\lambda - \tilde{\lambda})w_0\|_{L^{\frac{2n}{n-2}}(\Omega)} \\ &\leq C\|w_0\|_{L^{\frac{2n}{n-2}}(\Omega)} \\ &\leq C\|f\|_{B_{pp}^{2-1/p}(\partial\Omega)}. \end{aligned}$$

Thus because  $v_0 = w_0 + w_1$ , we have that

$$\|v_0\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C\|f\|_{B_{pp}^{2-1/p}(\partial\Omega)},$$

where the constant  $C$  is independent of  $\lambda$ , for large  $-\lambda$ . □

Next we examine the difference of the Dirichlet-to-Neumann maps, as  $\lambda \rightarrow -\infty$ .

**Proposition 3.3.** *Let  $p = 2n/(n+2)$  and  $\varepsilon > 0$ . Then*

$$(3.11) \quad \|\Lambda_{q_1}(\lambda) - \Lambda_{q_2}(\lambda)\|_{op} \rightarrow 0,$$

*when  $\lambda \rightarrow -\infty$  and  $0 < \varepsilon \ll 1$ , where  $\|\cdot\|_{op}$  denotes the operator norm on the space  $\mathcal{L}(B_{pp}^{2-1/p}(\partial\Omega), B_{pp}^{1-1/p-\varepsilon}(\partial\Omega))$ , of bounded linear operators.*

*Proof.* Let in  $u_j \in W^{2,p}(\Omega)$ ,  $j = 1, 2$  solve

$$(3.12) \quad \begin{aligned} (-\Delta + q_j - \lambda)u_j &= 0 \quad \text{in } \Omega, \\ \gamma u_j &= f \quad \text{on } \partial\Omega, \end{aligned}$$

with  $f \in B_{pp}^{2-1/p}(\partial\Omega)$ . Define  $u := u_1 - u_2$ . Then  $u$  will solve

$$(3.13) \quad \begin{aligned} (-\Delta + q_1 - \lambda)u &= (q_2 - q_1)u_2 \quad \text{in } \Omega, \\ \gamma u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Estimate (3.3) of Lemma 3.1, the Hölder inequality and the estimate (3.5) for non zero boundary conditions, gives us now that

$$(3.14) \quad \begin{aligned} \|u\|_{L^p(\Omega)} &\leq \frac{C}{|\lambda|} \|u_2\|_{L^{\frac{2n}{n-2}}(\Omega)} \|q_2 - q_1\|_{L^{\frac{n}{2}}(\Omega)} \\ &\leq \frac{C}{|\lambda|} \|f\|_{B_{pp}^{2-1/p}(\partial\Omega)}, \end{aligned}$$

where  $C$  is independent of  $\lambda$ , when  $-\lambda$  is large.

Estimate (3.4), the Hölder inequality and estimate (3.5) give us likewise that

$$(3.15) \quad \|u\|_{W^{2,p}(\Omega)} \leq C \|u_2\|_{L^{\frac{2n}{n-2}}(\Omega)} \|q_2 - q_1\|_{L^{\frac{n}{2}}(\Omega)} \leq C \|f\|_{B_{pp}^{2-1/p}(\partial\Omega)},$$

where in both estimates the constants  $C$  are independent of  $\lambda$ , when  $-\lambda$  is large.

We also need the basic interpolation property of Sobolev spaces according to which

$$(3.16) \quad \|u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1-\frac{s}{2}} \|u\|_{W^{2,p}(\Omega)}^{\frac{s}{2}},$$

for  $s \in (0, 2)$ .

From the definition of  $u$  it follows that

$$(3.17) \quad \|\Lambda_{q_1}(\lambda)f - \Lambda_{q_2}(\lambda)f\|_{B_{pp}^{1-1/p-\varepsilon}(\partial\Omega)} = \|\tilde{\gamma}u\|_{B_{pp}^{1-1/p-\varepsilon}(\partial\Omega)},$$

To estimate the right hand side of (3.17), we use the definition of the Neumann trace operator. By the continuity of the normal traces, property (3.16) and using estimates (3.14) and (3.15) we have that

$$\begin{aligned} \|\nu \cdot \nabla u|_{\partial\Omega}\|_{B_{pp}^{1-1/p-\varepsilon}(\partial\Omega)} &\leq C \|u\|_{W^{2-\varepsilon,p}(\Omega)} \\ &\leq C \|u\|_{L^p(\Omega)}^{\frac{\varepsilon}{2}} \|u\|_{W^{2,p}(\Omega)}^{1-\frac{\varepsilon}{2}} \\ &\leq \frac{C}{|\lambda|^{\frac{\varepsilon}{2}}} \|f\|_{B_{pp}^{2-1/p}(\partial\Omega)} \rightarrow 0, \end{aligned}$$

when  $\lambda \rightarrow -\infty$ . This together with (3.17) shows that the claim holds.  $\square$

We will prove Theorem 1.1, following the ideas in [17], [5] and [12]. We can view  $\Lambda(\lambda)f$  as a holomorphic function of  $\lambda$ , when  $\lambda \notin \text{Spec}(-\Delta + q)$ . The following Lemma will imply that  $[\Lambda_{q_1}(\lambda) - \Lambda_{q_2}(\lambda)]f$  is a polynomial in a half-plane  $\text{Re } \lambda \leq \lambda_0$ . The proof follows the argument in [12].

**Lemma 3.4.** *Suppose the assumptions of Theorem 1.1 hold and  $f \in B_{pp}^{2-1/p}(\partial\Omega)$ . For every  $m \in \mathbb{N}$ ,  $m > (n+4)/2$  and  $\lambda$ , s.t.  $-\lambda$  is large, we have that*

$$\frac{d^m}{d\lambda^m} [\Lambda_{q_1}(\lambda)f - \Lambda_{q_2}(\lambda)f] = 0.$$

*Proof.* Let  $\lambda \in \mathbb{C}$  s.t.  $\operatorname{Re} \lambda \leq \lambda_0$ , where  $\lambda_0$  is s.t. there is a unique solution  $u_q$  to the problem

$$(3.18) \quad \begin{aligned} (-\Delta + q - \lambda)u_q &= 0, \\ \gamma u_q &= f, \end{aligned}$$

for all  $\lambda$  with  $\operatorname{Re} \lambda \leq \lambda_0$ . The solution  $u_q$  can in general be expressed by means of the resolvent, by picking an extension  $Ef$ , of  $f$  to  $\Omega$  and then setting

$$(3.19) \quad u_q = Ef - R_q(\lambda)(-\Delta + q - \lambda)Ef.$$

Fix a  $\tilde{\lambda} \in \mathbb{R}$ , such that  $\tilde{\lambda} < \lambda$ . Consider an extension  $F \in W^{2,p}(\Omega)$  of  $f$  to  $\Omega$ , given by the solution to the problem

$$\begin{aligned} (-\Delta - \tilde{\lambda})F &= 0, \\ \gamma F &= f. \end{aligned}$$

Using (3.19) one can write  $u_q$  as

$$\begin{aligned} u_q &= F - R_q(\lambda)(-\Delta + q - \lambda)F \\ &= F - \sum_k \frac{1}{\lambda_k - \lambda} \langle (q + \tilde{\lambda} - \lambda)F, \varphi_k \rangle \varphi_k \end{aligned}$$

Now writing  $F$  as the series  $\sum_k (\varphi_k, F) \varphi_k$ , we get that

$$u_q = - \sum_k \frac{1}{\lambda_k - \lambda} \langle (q + \tilde{\lambda} - \lambda_k)F, \varphi_k \rangle \varphi_k.$$

Taking the derivative in  $\lambda$ , gives then

$$(3.20) \quad \frac{d^m}{d\lambda^m} u_q(\lambda) = -m! \sum_k \frac{1}{(\lambda_k - \lambda)^{m+1}} \langle (q + \tilde{\lambda} - \lambda_k)F, \varphi_k \rangle \varphi_k$$

The sum in (3.20) converges in  $L^2(\Omega)$  for every  $m \in \mathbb{N}$ . We need to show that it also converges in  $W^{2,p}(\Omega)$ , for large  $m$ .

Firstly by the Weyl law of Proposition 5.1 and estimate (5.2) we know that

$$\lambda_k \sim k^{2/n} \quad \text{and} \quad \|\varphi_k\|_{W^{2,p}(\Omega)} \leq C\lambda_k,$$

when  $k$  is large. Notice that  $|\lambda_k - \lambda| \gtrsim |\lambda_k|$ . We can estimate the Sobolev norm of the individual terms in the (3.20), for large  $k$  using

these observations in the following manner

$$\begin{aligned}
& \left\| \frac{1}{(\lambda_k - \lambda)^{m+1}} \langle (q + \tilde{\lambda} - \lambda_k)F, \varphi_k \rangle \varphi_k \right\|_{W^{2,p}(\Omega)} \\
& \leq C \frac{1}{|\lambda_k|^{m+1}} \|F\|_{L^{\frac{2n}{n-2}}(\Omega)} \|q + \tilde{\lambda} - \lambda_k\|_{L^{n/2}(\Omega)} \|\varphi_k\|_{W^{2,p}(\Omega)}^2 \\
& \leq C k^{\frac{-2(m+1)}{n}} k^{\frac{6}{n}}.
\end{aligned}$$

We need thus to choose  $m$ , so that  $m > (n+4)/2$  in order to make the series in (3.20) converge in the  $W^{2,p}(\Omega)$ -norm. It follows that

$$(3.21) \quad \tilde{\gamma} \frac{d^m}{d\lambda^m} u_q(\lambda) = -m! \sum_k \frac{1}{(\lambda_k - \lambda)^{m+1}} \langle (q + \tilde{\lambda} - \lambda_k)F, \varphi_k \rangle \tilde{\gamma} \varphi_k,$$

converges in the  $L^2(\Omega)$ -norm, when  $m$  is chosen large enough.

The claim follows from (3.21), since  $\tilde{\gamma} \varphi_{q_1,k} = \tilde{\gamma} \varphi_{q_2,k}$  and  $\lambda_{q_1,k} = \lambda_{q_2,k}$ , for every  $k$ , we have that  $\partial_\lambda^m [\Lambda_{q_1} - \Lambda_{q_2}](\lambda) f = 0$ , when  $m$  is large enough.  $\square$

The above proof shows furthermore that the function  $\lambda \mapsto \Lambda_q(\lambda)f$ , is holomorphic in a half-plane  $\operatorname{Re} \lambda \leq \lambda_0 \in (-\infty, 0)$ , for some  $\lambda_0$ , since by (3.21) we have a complex derivative  $\partial_\lambda^m \Lambda_q(\lambda)f = \nu \cdot \nabla \partial_\lambda^m u_q(\lambda)|_{\partial\Omega}$  exists, when  $m$  is large enough, from which it follows that it exists for every  $m \in \mathbb{N}$ .

Lemma 3.4 implies that  $[\Lambda_{q_1}(\lambda) - \Lambda_{q_2}(\lambda)]f$  is a polynomial in  $\lambda$ . Lemma 3.3 shows on the other hand that this polynomial goes to zero, as  $\lambda \rightarrow -\infty$ . It follows that the polynomial in question is zero, so that  $\Lambda_{q_1}(\lambda) = \Lambda_{q_2}(\lambda)$  for a fixed and large enough  $-\lambda$ . It is however known that the Dirichlet-to-Neumann map determines uniquely a potential  $q$  that is in  $L^{n/2}(\Omega)$  (see [14] and also [4], [8], [13]). It follows that  $q_1 = q_2$ , which proves Theorem 1.1.

#### 4. INCOMPLETE SPECTRAL DATA

In this section we consider the case of incomplete spectral data and prove Theorem 1.2 by adapting the ideas in [10] to case when  $q \in L^p(\Omega)$ , with  $p > n/2$ , if  $n = 3$  and  $p = n/2$ , if  $n \geq 4$ .

In the method used in [10] one considers non-real values of the spectral parameter  $\lambda$ . The arguments in the previous sections have dealt primarily with real  $\lambda$ . Our first task is therefore to prove a variant of Proposition 3.3 for certain complex values of  $\lambda$ . For our purposes it will be enough to consider  $\lambda$  in the set  $\mathcal{D}_s \subset \mathbb{C}$ ,  $s > 0$  defined as

$$(4.1) \quad \mathcal{D}_s := \mathbb{C} \setminus \left( \{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq s \frac{1}{2} (\operatorname{Im} \lambda)^2 - 1 \} \cup \operatorname{Spec}(-\Delta + q) \right).$$

**Lemma 4.1.** *Let  $f \in B_{pp}^{2-1/p}(\partial\Omega)$  and  $q_1, q_2 \in L^{n/2}(\Omega)$ . Then for  $\lambda \in \mathcal{D}_s$ ,  $s > 0$ , we have that*

$$\|\Lambda_{q_1}(\lambda)f - \Lambda_{q_2}(\lambda)f\|_{L^p(\partial\Omega)} \rightarrow 0,$$

as  $|\lambda| \rightarrow \infty$ .

*Proof.* Let  $\lambda \in \mathcal{D}_s$ . In the proof of Lemma 3.4 we derived the following expressions,

$$(4.2) \quad \frac{d^m}{d\lambda^m} u_q(\lambda) = -m! \sum_k \frac{1}{(\lambda_k - \lambda)^{m+1}} \langle (q + \tilde{\lambda} - \lambda_k) F, \varphi_k \rangle \varphi_k,$$

where  $u_q$  is a solution to (3.18). We can show that the sum converges in  $W^{2,p}(\Omega)$ , for large  $m$  in the same way we did in the proof of Lemma 3.4. This time we use that  $|\lambda_k - \lambda| \gtrsim |\lambda_k|^{1/2}$ , when  $\lambda \in \mathcal{D}_s$ , we get by estimating as in the proof of Lemma 3.4, that

$$\left\| \frac{1}{(\lambda_k - \lambda)^{j+1}} \langle (q + \tilde{\lambda} - \lambda_k) F, \varphi_k \rangle \varphi_k \right\|_{W^{2,p}(\Omega)} \leq C k^{\frac{-(m+1)}{n}} k^{\frac{6}{n}}.$$

We need thus to choose  $m$  so that  $m > n + 5$ , in order to make the series in (3.20) converge in the  $W^{2,p}(\Omega)$ -norm. It follows again that

$$(4.3) \quad \tilde{\gamma} \frac{d^m}{d\lambda^m} u_q(\lambda) = -m! \sum_k \frac{1}{(\lambda_k - \lambda)^{m+1}} \langle (q + \tilde{\lambda} - \lambda_k) F, \varphi_k \rangle \tilde{\gamma} \varphi_k,$$

converges in the  $L^2(\Omega)$ -norm, when  $m$  is chosen large enough.

We will rewrite equation (4.3), by integrating by parts as follows

$$\langle (q + \tilde{\lambda} - \lambda_k) F, \varphi_k \rangle \tilde{\gamma} \varphi_k = \int_{\partial\Omega} \nabla_n \varphi_k F dS \tilde{\gamma} \varphi_k =: A_{q,k}.$$

Equation (4.2) gives that

$$\frac{d^m}{d\lambda^m} \Lambda_q(\lambda) f = -m! \sum_k \frac{A_{q,k}}{(\lambda_k - \lambda)^{m+1}}.$$

when  $m$  is large and  $\lambda \in \mathcal{D}_s$ . Because the spectral data is identical for the operators  $-\Delta + q_j$ ,  $j = 1, 2$ , when  $k \geq k_0$ , we get that

$$\frac{d^m}{d\lambda^m} \left( \Lambda_{q_1}(\lambda) f - \Lambda_{q_2}(\lambda) f \right) = m! \sum_{k=1}^{k_0} \left( \frac{A_{q_2,k}}{(\lambda_{q_2,k} - \lambda)^{m+1}} - \frac{A_{q_1,k}}{(\lambda_{q_1,k} - \lambda)^{m+1}} \right).$$

By integrating  $m$ -times in  $\lambda$  we have

$$\Lambda_{q_1}(\lambda) f - \Lambda_{q_2}(\lambda) f = \sum_{k=1}^{k_0} \left( \frac{A_{q_2,k}}{\lambda_{q_2,k} - \lambda} - \frac{A_{q_1,k}}{\lambda_{q_1,k} - \lambda} \right) + \sum_{k=1}^{k_0} \lambda^{m-1} C_{q_1,q_2,k},$$

where  $C_{q_1,q_2,k} \in B_{pp}^{1-1/p}(\partial\Omega)$ . The left hand side will go to zero in the  $L^p(\partial\Omega)$ , when considering the special case  $\lambda \in \mathbb{R}$  and  $\lambda \rightarrow -\infty$  because of Proposition 3.3. The same applies to the first term on the right hand side. It follows that  $\sum_k C_{q_1,q_2,k} = 0$ . We hence see that for  $\lambda \in \mathcal{D}_s$

$$\|\Lambda_{q_1}(\lambda) f - \Lambda_{q_2}(\lambda) f\|_{L^p(\partial\Omega)} \rightarrow 0,$$

as  $|\lambda| \rightarrow \infty$ .

□

Following [10] we will consider

$$\varphi_{\lambda,\omega} := e^{i\sqrt{\lambda}\omega \cdot x},$$

where  $\lambda \in \mathbb{C} \setminus (0, \infty)$  and  $\omega \in S^{n-1}$ . Moreover we define

$$S(\lambda, \theta, \omega; q) := \int_{\partial\Omega} \Lambda_q(\lambda) \varphi_{\lambda,\omega} \varphi_{\lambda,-\theta} dS_x$$

Let  $R_q(\lambda)$  be the resolvent operator related to the Dirichlet problem (2.1). The following Lemma was established in [10] for  $q \in L^\infty(\Omega)$  (see Lemma 2.2 in [10]). The proof is essentially the same for  $q \in L^{n/2}(\Omega)$ .

**Lemma 4.2.** *We have the following identity,*

$$\begin{aligned} S(\lambda, \theta, \omega; q) &= \int_{\Omega} e^{-i\sqrt{\lambda}(\theta-\omega) \cdot x} q(x) dx - \frac{\lambda}{2}(\theta - \omega)^2 \int_{\Omega} e^{-i\sqrt{\lambda}(\theta-\omega) \cdot x} dx \\ &\quad - \langle R_q(\lambda)(q\varphi_{\lambda,\omega}), \overline{q\varphi_{\lambda,-\theta}} \rangle, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  is the duality pairing.

Our aim is to use the above Lemma to obtain the Fourier transform of the difference of the potentials. The first step is to choose the parameters  $\lambda$ ,  $\omega$  and  $\theta$  in a suitable way. More precisely we shall make the following choices in accordance with [10].

Let  $0 \neq \xi \in \mathbb{R}^n$  be fixed and  $\eta \in S^{n-1}$  and  $\xi \cdot \eta = 0$ . We will consider a specific  $\theta$  and  $\omega$  depending on a parameter  $m \in \mathbb{N}$ . These are chosen as follows

$$(4.4) \quad \begin{cases} \theta(m) &:= C(m)\eta + \xi/2m, \\ \omega(m) &:= C(m)\eta - \xi/2m, \\ \sqrt{\tau(m)} &:= m + i, \end{cases}$$

where  $C(m) := (1 - \frac{|\xi|^2}{4m^2})^{1/2}$ , so that  $\theta(m), \omega(m) \in S^{n-1}$ . It follows that

$$\begin{cases} \sqrt{\tau(m)}(\theta(m) - \omega(m)) \rightarrow \xi, \\ \text{Im } \tau(m) \rightarrow \infty, \\ \text{Im } \sqrt{\tau(m)}\theta(m), \text{Im } \sqrt{\tau(m)}\omega(m) \leq C < \infty, \end{cases}$$

as  $m \rightarrow \infty$ . We will furthermore use the abbreviations

$$\psi_\omega := \varphi_{\tau(m), \omega(m)} \quad \text{and} \quad \psi_\theta := \varphi_{\tau(m), -\theta(m)}.$$

Lemma 4.1 implies now the following.

**Lemma 4.3.** *We have that*

$$S(\tau, \theta, \omega; q_1) - S(\tau, \theta, \omega; q_2) \rightarrow 0,$$

as  $m \rightarrow \infty$ .

*Proof.* Using the definition of  $S$  we see that we need to show that

$$\int_{\partial\Omega} (\Lambda_{q_1}(\tau)\psi_\omega - \Lambda_{q_2}(\tau)\psi_\omega)\psi_\theta dS_x \rightarrow 0,$$

Where  $\tau = \tau(m)$  and  $s > 0$ , are such that  $\tau(m) \in \mathcal{D}_s$ . In addition we have that  $\|\psi_\theta\|_{L^\infty} = \|\varphi_{\tau(m), -\theta(m)}\|_{L^\infty} \leq C < \infty$ , when  $m \rightarrow \infty$ . The claim follows now from Lemma 4.1, by the Hölder inequality.  $\square$

Lemmas 4.2 and 4.3 imply that

$$(4.5) \quad \int_{\Omega} e^{-i\sqrt{\tau}(\theta-\omega)\cdot x} (q_1 - q_2) + \sum_{j=1,2} (-1)^j \langle q_j R_{q_j}(\tau)(q_j \psi_\omega), \overline{\psi_\theta} \rangle \rightarrow 0,$$

as  $m \rightarrow \infty$ , when  $\theta, \tau$  and  $\omega$  is chosen as in (4.4). If we can now show that the two terms containing the Dirichlet resolvents  $R_{q_j}$  vanish in the limit, we then obtain that

$$\int_{\Omega} e^{i\xi\cdot x} (q_1 - q_2) dx = 0,$$

and thus that  $q_1 = q_2$ , which proves Theorem 1.2. It remains therefore to analyze the terms in (4.5) containing the resolvents. To this end we derive the following resolvent estimate.

**Remark 4.4.** *We can assume for simplicity that the operators  $(-\Delta + q_j)$  are positive in the sense that*

$$(4.6) \quad C\|u\|_{H^1(\Omega)}^2 \leq \langle (-\Delta + q_j)u, u \rangle,$$

*when  $u \in H_0^1(\Omega)$ , since the above inequality holds for  $(-\Delta + q_j + \lambda_0)$ , where  $\lambda_0$  is a suitable constant. This can be seen by using (5.1). Adding the  $\lambda_0$ , only shifts the spectrum of the operators  $(-\Delta + q_j)$ , so that the spectral data for  $(-\Delta + q_j + \lambda_0)$  coincide for  $j = 1, 2$ , provided that the spectral data for  $(-\Delta + q_j)$ , for  $j = 1, 2$  coincide.*

**Lemma 4.5.** *Assume that  $q \in L^{n/2}(\Omega, \mathbb{R})$  is such that (4.6) holds, and that  $\tau(m) = (m + i)^2$ . Then for  $f \in L^p(\Omega)$ ,  $p = 2n/(n + 2)$ , we have that*

$$(4.7) \quad \|R_q(\tau(m))f\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C |\operatorname{Im} \tau(m)| \|f\|_{L^{\frac{2n}{n+2}}(\Omega)},$$

*where  $C$  is independent of  $\tau(m)$ .*

*Proof.* By (4.6) we have that

$$(4.8) \quad C\|u\|_{H^1(\Omega)}^2 \leq \langle (-\Delta + q)u, u \rangle = \left\langle \sum_k \lambda_k \langle \varphi_k, u \rangle \varphi_k, u \right\rangle,$$



which holds for  $u \in H_0^1(\Omega)$ . Let  $f \in C^\infty(\Omega)$ . By (5.4) we know that the resolvent can be expressed as the sum

$$(4.9) \quad R_q(\lambda)f = \sum_{k=1}^{\infty} \frac{\langle \varphi_k, f \rangle}{\lambda_k - \lambda} \varphi_k,$$

that is convergent in the  $L^2(\Omega)$ -norm. Using this and (4.8) we get that

$$(4.10) \quad \begin{aligned} \|R_q(\lambda)f\|_{H^1(\Omega)}^2 &\leq \sum_{j=1}^{\infty} \lambda_j \left| \left( \varphi_j, \sum_{k=1}^{\infty} \frac{\langle \varphi_k, f \rangle}{\lambda_k - \lambda} \varphi_k \right) \right|_{L^2}^2 \\ &\leq C \sum_{k=1}^{\infty} \lambda_k \frac{|\langle \varphi_k, f \rangle|^2}{|\lambda_k - \lambda|^2} \\ &\leq C \sup_k \left| \frac{\lambda_k}{\lambda_k - \lambda} \right|^2 \sum_{k=1}^{\infty} \frac{|\langle \varphi_k, f \rangle|^2}{\lambda_k}. \end{aligned}$$

Taking the Fourier representation and using (5.4) we get that

$$\langle f, R_q(0)f \rangle = \sum_{j,k} \left( \langle \varphi_j, f \rangle \varphi_j, \frac{\langle \varphi_k, f \rangle}{\lambda_k} \varphi_k \right) = \sum_{k=1}^{\infty} \frac{|\langle \varphi_k, f \rangle|^2}{\lambda_k}.$$

So that by the continuity of  $R_q(0)$  we have that

$$(4.11) \quad \begin{aligned} \sum_{k=1}^{\infty} \frac{|\langle \varphi_k, f \rangle|^2}{\lambda_k} &\leq C \|f\|_{H^{-1}(\Omega)} \|R_q(0)f\|_{H_0^1(\Omega)} \\ &\leq C \|f\|_{H^{-1}(\Omega)}^2. \end{aligned}$$

Combining (4.10) and (4.11), gives then

$$(4.12) \quad \|R_q(\lambda)f\|_{H^1(\Omega)}^2 \leq C \sup_k \left| \frac{\lambda_k}{\lambda_k - \lambda} \right|^2 \|f\|_{H^{-1}(\Omega)}^2,$$

for  $f \in C^\infty(\Omega)$ , where  $C$  does not depend on  $\lambda$ . Using a density argument shows that this holds also when  $f \in H^{-1}(\Omega)$ .

Assume now that  $\lambda = \tau(m) = (m + i)^2$  and consider the expression

$$\sup_k \left| \frac{\lambda_k}{\lambda_k - \tau(m)} \right| \leq C \sup_k \frac{|\lambda_k|}{|\lambda_k - m^2 + 1| + |2mi|}.$$

The function  $f(x) = x/(|x - m^2 + 1| + |2m|)$  is monotonely decreasing for  $x \geq m^2 - 1$  and  $f(m^2 - 1) \leq m$ . For  $0 < x \leq m^2 - 1$  we have that  $f(x) \leq (m^2 - 1)/m \lesssim m$ . Since  $\lambda_k > 0$  we have the estimate

$$\sup_k \left| \frac{\lambda_k}{\lambda_k - \tau(m)} \right| \leq C |\operatorname{Im} \tau(m)|.$$

The claim follows now from (4.12), by setting  $\lambda = \tau(m)$  and using Sobolev embedding.

□

**Remark 4.6.** *One would expect that the above  $L^2$ -theory based estimate could be improved, since it is well known that in the case of  $\Omega = \mathbb{R}^n$ , or when  $\Omega$  is a Riemannian manifold without boundary, and  $q = 0$  one has so called "uniform Sobolev estimates", see e.g. [11] and [8]. Estimates like Proposition 2.5 also seem suggest that there is room for improvement. We are however not aware of any such uniform estimates for the Dirichlet resolvent for domains with a boundary which would include  $\tau(m)$ , when  $m$  is large.*

We are now ready to show that the resolvent terms in (4.5) vanish.

**Lemma 4.7.** *Assume that  $q \in L^p(\Omega, \mathbb{R})$ , with  $p = n/2$ , if  $n \geq 4$  and  $p > n/2$ , if  $n = 3$ , then we have that*

$$\langle qR_q(\tau(m))(q\psi_\omega), \overline{\psi_\theta} \rangle \rightarrow 0,$$

as  $m \rightarrow \infty$ .

*Proof.* It is enough to show that the  $L^1$ -norm of the first term in the duality pairing on the left hand side of the claim goes to zero, since  $\varphi_{\tau(m), -\theta(m)} < C < \infty$ , where  $C$  can be picked to be independent of  $m$ .

By the Hölder inequality, we have that

$$(4.13) \quad \|qR_q(\tau(m))(q\psi_\omega)\|_{L^1(\Omega)} \leq \|q\|_{L^p(\Omega)} \|R_q(\tau(m))(q\psi_\omega)\|_{L^{p^*}(\Omega)}.$$

It will be convenient to write  $p$ , as  $p = \frac{n+\epsilon}{2}$ , for some  $\epsilon > 0$  and the Hölder conjugate  $p^*$ , as  $p^* = \frac{n+\epsilon}{n+\epsilon-2}$ .

Suppose firstly that  $n \geq 4$ . In this case  $p^* \leq 2 \leq p$ . Estimate (5.5) gives us immediately that

$$(4.14) \quad \|R_q(\tau(m))(q\psi_\omega)\|_{L^{\frac{n+\epsilon}{n+\epsilon-2}}(\Omega)} \leq \frac{C}{|\operatorname{Im} \tau(m)|} \|q\psi_\omega\|_{L^{\frac{n+\epsilon}{2}}(\Omega)} \rightarrow 0,$$

as  $m \rightarrow \infty$ . Notice that this is true even when  $\epsilon = 0$ , proving the claim when  $n \geq 4$ .

Assume now that  $n = 3$ . In this case we use the Riesz-Thorin interpolation theorem to obtain an estimate for the  $L^{p^*}$ -norm above. The Riesz-Thorin interpolation theorem states that

$$\|T\varphi\|_{L^{q_\theta}} \leq C_0^\theta C_1^{1-\theta} \|\varphi\|_{L^{p_\theta}},$$

where  $p_\theta^{-1} = \theta p_0^{-1} + (1-\theta)p_1^{-1}$  and  $q_\theta^{-1} = \theta q_0^{-1} + (1-\theta)q_1^{-1}$ , provided we have the estimates  $\|T\varphi\|_{L^{q_j}} \leq C_j \|\varphi\|_{L^{p_j}}$ , for  $j = 0, 1$ . By Lemma 4.5 and (5.5) we have that

$$\begin{aligned} \|R_q(\tau(m))f\|_{L^2(\Omega)} &\leq \frac{C}{|\operatorname{Im} \tau(m)|} \|f\|_{L^2(\Omega)}, \\ \|R_q(\tau(m))f\|_{L^{\frac{2n}{n-2}}(\Omega)} &\leq C |\operatorname{Im} \tau(m)| \|f\|_{L^{\frac{2n}{n+2}}(\Omega)}. \end{aligned}$$

Applying the Riesz-Thorin interpolation Theorem to these estimates and taking  $\theta \in (0, 1)$  to be  $\theta = \frac{3-3\epsilon}{6+2\epsilon}$ , gives that

$$\|R_q(\tau(m))(q\psi_\omega)\|_{L^{\frac{n+\epsilon}{n+\epsilon-2}}(\Omega)} \leq C |\operatorname{Im} \tau(m)|^{2\theta-1} \|q\psi_\omega\|_{L^{\frac{n+\epsilon}{2}}(\Omega)} \rightarrow 0,$$

as  $m \rightarrow \infty$ , since  $\theta < 1/2$ . This together with (4.14) and (4.13) shows that

$$\langle qR_q(\tau(m))(q\psi_\omega), \overline{\psi_\theta} \rangle \rightarrow 0,$$

as  $m \rightarrow \infty$ . □

## 5. APPENDIX A. THE SPECTRUM

In this section we review some basic facts from the spectral theory relating to the operator  $-\Delta + q$ , with  $q \in L^{n/2}(\Omega, \mathbb{R})$ . A weak solution  $u \in H_0^1(\Omega)$  to (2.8) is a function for which

$$\Phi(u, v) := \int_{\Omega} \nabla u \cdot \nabla \bar{v} + qu\bar{v} = \int_{\Omega} F\bar{v},$$

holds for every  $v \in H_0^1(\Omega)$ .

The sesquilinear form  $\Phi : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{C}$  is continuous, since by the Hölder inequality and Sobolev embedding we have that

$$\begin{aligned} |\Phi(u, v)| &\leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + \|q\|_{L^{n/2}(\Omega)} \|u\|_{L^{\frac{2n}{n-2}}(\Omega)} \|v\|_{L^{\frac{2n}{n-2}}(\Omega)} \\ &\leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned}$$

The form  $\Phi$  is in addition coercive, which can be seen as follows. Let  $q_k \in C^\infty(\Omega)$ , be such that  $q_k \rightarrow q$ , in the  $L^{n/2}(\Omega)$ -norm. Then

$$\begin{aligned} \Phi(u, u) &\geq \|u\|_{H^1(\Omega)}^2 - \|q_k\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2 - \|q - q_k\|_{L^{n/2}(\Omega)} \|u\|_{L^{\frac{2n}{n-2}}(\Omega)}^2 \\ (5.1) \quad &\geq C \|u\|_{H^1(\Omega)}^2 - C_0 \|u\|_{L^2(\Omega)}^2. \end{aligned}$$

The operator  $L := -\Delta + q : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  can be understood as the operator given by  $\langle Lu, v \rangle = \Phi(u, v)$ , where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing. It follows that  $L$  is also coercive and continuous. The adjoint  $L^*$  of  $L$  may be defined as  $\langle L^*u, v \rangle = \overline{\Phi(u, v)}$ . It follows then that  $L$  is self-adjoint on  $H_0^1(\Omega)$ .

We have moreover that  $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ , where  $L^2(\Omega)$  is called the pivot space for  $H_0^1(\Omega)$ . Since  $L : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  is bounded, coercive and self-adjoint, and  $L^2(\Omega)$  is a pivot space, we have by Theorem 2.37 in [16] firstly that

There is sequence of eigenfunctions  $\varphi_k \in H_0^1(\Omega)$  and corresponding eigenvalues  $-\infty < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow \infty$ .

As a second consequence of Theorem 2.37 in [16] is that

The set  $\{\varphi_k\}$  is a complete orthonormal basis in  $L^2(\Omega)$ .

The Sobolev norm of the eigenfunctions also have nice estimates. By the estimate of Proposition 2.1 we have that

$$(5.2) \quad \|\varphi_k\|_{W^{2,p}(\Omega)} \leq C_0(|\lambda_k| + 1)\|\varphi_k\|_{L^p(\Omega)}.$$

For  $\lambda \in \mathbb{C}$ , s.t.  $\lambda \notin \{\lambda_k\} = \text{Spec}(-\Delta + q)$  we have that the Dirichlet resolvent, i.e. the operator  $R_q(\lambda) := (-\Delta + q - \lambda)^{-1} : H^{-1}(\Omega) \rightarrow H_0^1(\Omega)$  is continuous. Hence we have the following estimate

$$(5.3) \quad \|R_q(\lambda)f\|_{H^1(\Omega)} \leq C_\lambda \|f\|_{H^{-1}(\Omega)}.$$

The resolvent can be expressed as the sum

$$(5.4) \quad R_q(\lambda)f = \sum_{k=1}^{\infty} \frac{\langle \varphi_k, f \rangle}{\lambda_k - \lambda} \varphi_k,$$

which is convergent in the  $L^2(\Omega)$ -norm (see for instance Corollary 2.39 in [16]). From this one can furthermore derive the norm estimate

$$(5.5) \quad \|R_q(\lambda)f\|_{L^2(\Omega)} \leq \frac{C}{|\text{Im } \lambda|} \|f\|_{L^2(\Omega)}.$$

A further fact we need concerning the spectrum of the operator  $-\Delta + q$  is the following Weyl law, that pertains to potentials  $q \in L^{n/2}(\Omega, \mathbb{R})$ .

**Proposition 5.1.** *Let  $q \in L^{n/2}(\Omega, \mathbb{R})$ ,  $n \geq 3$ . Then for the Schrödinger operator  $-\Delta + q$ , with form domain  $H_0^1(\Omega)$ , we have the Weyl law*

$$\lambda_k \sim \frac{4\pi^2}{(V_B V_\Omega)^{2/n}} k^{2/n},$$

as  $k \rightarrow \infty$ , where  $V_B$  is the volume of the  $n$ -dimensional unit ball and  $V_\Omega$  is the volume of  $\Omega$ .

*Proof.* Denote the Dirichlet eigenvalues for the Laplacian (i.e. when  $q = 0$ ), by  $\lambda_1^D \leq \lambda_2^D \leq \dots$ . These exhibit the following Weyl law,

$$\lambda_k^D \sim \frac{4\pi^2}{(V_B V_\Omega)^{2/n}} k^{2/n},$$

as  $k \rightarrow \infty$  (see e.g. Sect. 13.4 of [22]).

Next, the mini-max principle (see e.g. Sect. 4.5 in [7]) says that

$$\lambda_k = \min_{\substack{X \subseteq D \\ \dim X = k}} \max_{\substack{\varphi \in X \\ \|\varphi\|=1}} \int_{\Omega} (|\nabla \varphi|^2 + q |\varphi|^2)$$

and

$$\lambda_k^D = \min_{\substack{X \subseteq D \\ \dim X = k}} \max_{\substack{\varphi \in X \\ \|\varphi\|=1}} \int_{\Omega} |\nabla \varphi|^2,$$

where the minima are to be taken over vector subspaces  $X$  of  $D$ . By Lemma 5.2 there exists a constant  $C_\varepsilon \in \mathbb{R}_+$ , for  $\varepsilon > 0$ , such that

$$(5.6) \quad \int_{\Omega} q |\varphi|^2 \leq \varepsilon \|\nabla \varphi\|_{L^2(\Omega)}^2 + C_\varepsilon \|\varphi\|_{L^2(\Omega)}^2,$$

for  $\varphi \in D$ . Combining this relative bound with the mini-max principle gives

$$\begin{aligned} \lambda_k &= \min_{\substack{X \subseteq D \\ \dim X = k}} \max_{\substack{\varphi \in X \\ \|\varphi\|=1}} \int_{\Omega} (|\nabla \varphi|^2 + q |\varphi|^2) \\ &\leq \min_{\substack{X \subseteq D \\ \dim X = k}} \max_{\substack{\varphi \in X \\ \|\varphi\|=1}} \int_{\Omega} ((1 + \varepsilon) |\nabla \varphi|^2 + C_\varepsilon |\varphi|^2) = (1 + \varepsilon) \lambda_k^D + C_\varepsilon. \end{aligned}$$

Similarly, we obtain

$$\lambda_k \geq (1 - \varepsilon) \lambda_k^D - C_\varepsilon.$$

Thus,

$$(1 - 2\varepsilon) \lambda_k^D \leq \lambda_k \leq (1 + 2\varepsilon) \lambda_k^D$$

for sufficiently large  $k \in \mathbb{Z}_+$ . Since  $\varepsilon$  was arbitrarily small, we have established that  $\lambda_k \sim \lambda_k^D$ .  $\square$

We need to justify the use of (5.6) in the previous proof, in order to finish it. This is done in the following Lemma.

**Lemma 5.2.** *Let  $q \in L^{n/2}(\Omega, \mathbb{R})$ ,  $n \geq 3$  and  $\varphi \in H_0^1(\Omega)$ . Then for  $\varepsilon > 0$ , we have*

$$\int_{\Omega} q \varphi^2 \leq \varepsilon \|\nabla \varphi\|_{L^2(\Omega)}^2 + C_\varepsilon \|\varphi\|_{L^2(\Omega)}^2.$$

*Proof.* By the Hölder inequality we have that

$$\int_{\Omega} q \varphi^2 \leq \|q \varphi\|_{L^{\frac{2n}{n+2}}(\Omega)} \|\varphi\|_{L^{\frac{2n}{n-2}}(\Omega)}.$$

By the Poincaré inequality and Sobolev embedding we have that

$$\begin{aligned} \int_{\Omega} q \varphi^2 &\leq C \|q \varphi\|_{L^{\frac{2n}{n+2}}(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \\ (5.7) \quad &\leq \frac{C}{\varepsilon} \|q \varphi\|_{L^{\frac{2n}{n+2}}(\Omega)}^2 + \varepsilon \|\nabla \varphi\|_{L^2(\Omega)}^2. \end{aligned}$$

Next we pick a smooth approximation  $q_k \in C^\infty(\Omega)$ , s.t.  $q_k \rightarrow q$ , in  $L^{n/2}(\Omega)$ . We have that

$$\begin{aligned} \|q \varphi\|_{L^{\frac{2n}{n+2}}(\Omega)}^2 &\leq 2 \|q_k \varphi\|_{L^{\frac{2n}{n+2}}(\Omega)}^2 + 2 \|(q - q_k) \varphi\|_{L^{\frac{2n}{n+2}}(\Omega)}^2 \\ &\leq 2 \|q_k\|_{L^\infty(\Omega)}^2 \|\varphi\|_{L^2(\Omega)}^2 + 2 \|(q - q_k)\|_{L^{\frac{n}{2}}(\Omega)}^2 \|\varphi\|_{L^{\frac{2n}{n-2}}(\Omega)}^2 \\ &\leq 2 \|q_k\|_{L^\infty(\Omega)}^2 \|\varphi\|_{L^2(\Omega)}^2 + 2 \|q - q_k\|_{L^{\frac{n}{2}}(\Omega)}^2 \|\nabla \varphi\|_{L^2(\Omega)}^2. \end{aligned}$$

Using this with estimate (5.7) and picking  $\|q - q_k\|_{L^{\frac{n}{2}}}$  to be small in a suitable way gives

$$\int_{\Omega} q \varphi^2 \leq \varepsilon \|\nabla \varphi\|_{L^2(\Omega)}^2 + C_\varepsilon \|\varphi\|_{L^2(\Omega)}^2.$$

$\square$

## 6. APPENDIX B. BESOV SPACES

Here we review some basic definitions and properties of the Besov spaces that we use. The main reason for considering Besov spaces is that they give a more precise meaning to the restriction  $u|_{\partial\Omega}$ , when  $u$  is a member of the Sobolev space  $W^{s,p}(\Omega)$ . The main reference for this section is [21].

We use the following definitions of the spaces  $B_{pp}^s(\Omega)$ . We split  $s \in \mathbb{R}$ , as  $s = [s] + \{s\}$ , where  $[s]$  is an integer and  $0 \leq \{s\} < 1$ . The space  $B_{pp}^s(\mathbb{R}^n)$ , with  $1 < p < \infty$  and  $0 < s \notin \mathbb{N}$ , consists of those functions in  $f \in L^p(\mathbb{R}^n)$ , for which the norm

$$\|f\|_{B_{pp}^s(\mathbb{R}^n)} := \|f\|_{W^{[s],p}(\mathbb{R}^n)} + \sum_{|\alpha|=[s]} \left( \int_{\mathbb{R}^n} \frac{\|D^\alpha f(\cdot + y) - D^\alpha f\|_{L^p(\mathbb{R}^n)}^p}{|y|^{n+p\{s\}}} dy \right)^{1/p},$$

is finite, where  $\alpha := (\alpha_1, \dots, \alpha_n)$  is a multi-index and  $D^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ . For a bounded domain  $\Omega \subset \mathbb{R}^n$ , we define the space  $B_{pp}^s(\Omega)$  as the set

$$B_{pp}^s(\Omega) := \{f \in L^p(\Omega) : \exists g \in B_{pp}^s(\mathbb{R}^n) \text{ s.t. } g|_\Omega = f\},$$

equipped with the norm

$$\|f\|_{B_{pp}^s(\Omega)} := \inf \{\|g\|_{B_{pp}^s(\mathbb{R}^n)} : g \in B_{pp}^s(\mathbb{R}^n) \text{ s.t. } g|_\Omega = f\}.$$

Our main interest in the spaces  $B_{pp}^s(\Omega)$  is that they provide natural trace spaces for the spaces  $W^{2,p}(\Omega)$ . The Dirichlet trace operator  $\gamma$ , may be defined by

$$\gamma u := u|_{\partial\Omega}.$$

for  $u \in C^\infty(\Omega)$ . One can then extend this map to a continuous map  $\gamma : W^{2,p}(\Omega) \rightarrow B_{pp}^{2-1/p}(\partial\Omega)$ , so that<sup>1</sup>

$$\|\gamma u\|_{B_{pp}^{2-1/p}(\partial\Omega)} \leq C \|u\|_{W^{2,p}(\Omega)}.$$

We will use the notation  $u|_{\partial\Omega} := \gamma u$ . The Dirichlet trace operator  $\gamma$  has moreover a right inverse  $E$ ,  $E : B_{pp}^{2-1/p}(\partial\Omega) \rightarrow W^{2,p}(\Omega)$ , for which  $\gamma E f = f$  and

$$\|E f\|_{W^{2,p}(\Omega)} \leq C \|f\|_{B_{pp}^{2-1/p}(\partial\Omega)},$$

See section 3.3.3. in [21].

Likewise one can define the Neumann trace operator  $\tilde{\gamma}$  given by

$$\tilde{\gamma} u := \partial_\nu u|_{\partial\Omega}$$

---

<sup>1</sup> Firstly we have that  $W^{2,p}(\Omega) = F_{p2}^2(\Omega)$ , where  $F_{p2}^2(\Omega)$  is a Triebel space. See [21], section 3.4.2, p. 208. By Theorem 3.3.3 in [21], we have on the other hand that

$$\gamma : F_{p2}^2(\Omega) \rightarrow B_{pp}^{2-1/p}(\partial\Omega),$$

is continuous

for  $u \in C^\infty(\Omega)$  and where  $\nu$  is the outer unit normal vector to  $\partial\Omega$ . The Neumann trace operator

$$\tilde{\gamma}: W^{2,p}(\Omega) \rightarrow B_{pp}^{1-1/p}(\partial\Omega),$$

is bounded and linear, which follows similarly as the continuity for the Dirichlet trace operator. For  $u \in W^{2,p}(\Omega)$ , we will use for  $\tilde{\gamma}u$  the notation  $\partial_\nu u|_{\partial\Omega}$ .

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